Parameter and State Estimation of Vibrating Structures Equipped with Geometrically Consistent Tuned Mass Damper

A Thesis Phase-I Interim Evaluation Report Submitted in Partial Fulfillment of the Requirements for the Degree of

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by

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<u>CERTIFICATE</u>

*This is to certify that the work contained in this thesis entitled “**Parameter and State Estimation of Vibrating Structures Equipped with Geometrically Consistent Tuned Mass Damper**” is a bonafide work of **Roshan Kumar (Roll No. 200104093**), carried out in the Department of Civil Engineering, Indian Institute of Technology Guwahati under my supervision and that it has not been submitted elsewhere for a degree.*

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Chapter 1

Introduction

In the realm of structural dynamics, the investigation of vibrating systems has emerged as a pivotal field, providing essential insights into the understanding of structural behavior and resilience against external forces. The intricacies of these systems necessitate a sophisticated modeling approach, particularly when dealing with structures equipped with geometrically consistent tuned mass dampers (GTMDs). This research ventures into the challenging domain of estimating parameters and states of vibrating structures adorned with GTMDs with precision.

In the contemporary era, the application of dynamical systems has become increasingly relevant, serving as a lens through which we analyze both natural phenomena, such as seismic events, and technological innovations. Traditional modeling often relies on Euclidean spaces; however, the inherent complexity of certain systems demands a nuanced approach—one that incorporates the geometry within which these structures evolve. Managing kinematic constraints becomes paramount, and this work resides at the intersection of applied mathematics, theoretical sciences, and mechanical control, navigating the intricate landscape of dynamical systems on Riemannian manifolds.

The study embarks on a mission to unravel the dynamics of vibrating structures with GTMDs, an area of profound significance in earthquake engineering and structural design.

Drawing inspiration from the complexity of these systems, the thesis employs Kalman filters as a foundational tool for parameter and state estimation on Riemannian manifolds.

While existing literature has made substantial contributions to the understanding of dynamical systems on manifolds, certain aspects, particularly related to parameter and state estimation in the context of vibrating structures with GTMDs, remain underexplored. This work seeks to bridge this gap by employing Kalman filters to refine the estimation process, emphasizing their application in the realm of Lie Algebra and Riemannian manifolds. Comparative analyses with existing numerical schemes further validate the effectiveness of the proposed approaches.

By unraveling the intricacies of parameter and state estimation in vibrating structures equipped with GTMDs on Riemannian manifolds, this thesis contributes to the broader understanding of structural dynamics. The incorporation of Kalman filters within the context of Lie Algebra not only enhances the accuracy of estimation but also paves the way for robust methodologies, thereby advancing the field and offering novel perspectives on the analysis of such dynamic systems.

1.1 Introduction to Key Concepts and Literature Survey

Structural vibrations pose significant challenges in the field of civil engineering, requiring innovative solutions to mitigate their effects on the integrity and safety of buildings and other structures. One such solution is the application of Tuned Mass Dampers (TMDs), with a specific focus on Pendulum Tuned Mass Dampers (PTMDs). This literature review explores the state of the art in modal parameter estimation of structures equipped with PTMDs, along with advancements in the field of Riemannian manifolds and their potential applications in structural dynamics.

Pendulum tuned mass dampers (PTMDs) have emerged as a prominent solution for mitigating structural vibrations induced primarily by dynamic forces such as wind. These devices have found practical application in various full-scale structures, significantly enhancing their ability to withstand environmental forces. The works of A. J. Roffel and Sriram Narasimhan [1] have addressed the need for comprehensive methodologies to estimate modal parameters while PTMDs are in service. In their paper, they present a method for time domain modal parametric identification of natural frequencies, mode shapes, and modal damping ratios of structures equipped with PTMDs. This research acknowledges the crucial role of PTMDs in structural dynamics, emphasizing their adaptive passive nature with mechanisms to adjust auxiliary frequency and damping.

One key observation from the studies of Roffel and Narasimhan [1] is the inherent uncertainty associated with estimating the first modal damping. This uncertainty partly stems from the frequency-dependent behavior of the dampers. Interestingly, this uncertainty is notably reduced for the second mode, suggesting that the damper exhibits less frequency-dependent behavior at higher frequencies. This insight underscores the complexity of PTMDs and highlights the importance of a precise and robust estimation framework for modal parameters, especially in cases involving varying dynamic conditions.

Parallel to the advancements in structural dynamics, the field of differential geometry has gained attention in modeling non-linearities by confining parts of the model to Riemannian manifolds. The work of Søren Hauberg, François Lauze, and Kim Steenstrup Pedersen[2] introduces a novel algorithm that generalizes the unscented transform and the unscented Kalman filter for Riemannian manifolds. This pioneering research provides a generic optimization framework for these domains and demonstrates its robustness and convergence across various applications. In particular, the Riemannian unscented Kalman filter (UKF) is noted for producing smoother motion estimates, making it a promising tool for modeling complex structural vibrations with improved accuracy.

Building on this, the paper by Tripura, Panda, and Hazra [3], extends the horizon of real-time modal identification techniques. Their work introduces a novel approach that leverages first-order error-adapted eigen perturbation to enhance the accuracy and efficiency of real-time modal identification in vibrating structures. By incorporating differential geometry concepts from Pennec [4] into the identification process, this research represents an innovative step towards addressing the challenges posed by dynamic structural behavior.

1.2 Objective

The Study in this paper is carried out with the following objectives in mind

1. State and parameter estimation of systems evolving on Riemannian Manifolds, Eg: Chaotic Pendulum (3D Pendulum which has large swing angle) and Passive and Semi-Active TMD (Tuned Mass Dampers).

2. Optimisation and Parameter estimation of Pendulum Cart System evolving on SO(3) Manifold.

3. Mathematically model Geometrically consistent pendulum should capture all the displacements and rotations (including 3 translations, 3 rotations, and interaction coupled displacement).

1.3 Data and Methods

In this section, we outline the data generation process for simulating structural vibrations equipped with geometrically consistent tuned mass dampers (TMDs) over a manifold using the Geometric Ito-Taylor 1.5 method introduced by Panda and Hazra [5]. Additionally, we provide the dynamic equation for a chaotic pendulum in the Special Orthogonal Group SO(3) manifold, which serves as the foundation for our simulations.

1.3.1 Data Generation Using Geometric Ito-Taylor 1.5 Method

The Geometric Ito-Taylor 1.5 method is a powerful numerical technique for simulating stochastic differential equations, making it well-suited for modeling dynamic systems with uncertainty. In our study, we employ this method to generate simulated data representing the behavior of structures equipped with geometrically consistent tuned mass dampers. The following steps outline the data generation process:

Dynamic Equation for Chaotic Pendulum on SO(3) Manifold

To simulate the behavior of a structure with a chaotic pendulum equipped with a geometrically consistent tuned mass damper, we first need to establish the dynamic equation governing the system. The dynamic equation for a chaotic pendulum on the SO(3) manifold can be represented as follows:

$$I \bullet \dot{\omega} = -\omega * I \bullet \omega + u \tag{1.1}$$

Where:

- I is the moment of inertia matrix.
- ω represents the angular velocity vector.
- $\dot{\omega}$ is the time derivative of the angular velocity.
- u represents the control input, which incorporates the effects of the tuned mass damper and any external forces or disturbances.

Numerical Integration with Geometric Ito-Taylor 1.5

We use the geometric Ito-Taylor 1.5 method to numerically integrate the dynamic equation over time. This method provides an accurate representation of the system's behavior, accounting for both deterministic and stochastic components. The numerical integration process involves discretizing the time domain and updating the state variables (angular velocity and orientation) at each time step.

Incorporating Stochastic Elements

To account for uncertainties and external disturbances in real-world scenarios, we introduce stochastic elements into our simulations. The geometric Ito-Taylor 1.5 method allows us to model stochastic processes effectively. We include these stochastic components in the control input to simulate the effects of uncertain forces or disturbances.

Parameterization of the PTMD

To achieve geometric consistency with the tuned mass damper, we parameterize the PTMD's characteristics within the control input u. This includes adjusting the auxiliary frequency and damping properties of the PTMD as per the design requirements.

Simulation Output

The output of the simulation provides time-series data representing the structural vibrations of the system, including the motion of the pendulum, the response of the PTMD, and any other relevant parameters of interest.

By following this data generation process, we can simulate the behavior of structures equipped with geometrically consistent tuned mass dampers under various conditions, allowing us to investigate modal parameter estimation and structural response in a controlled and repeatable manner.

Chapter 2

Background

In this chapter, we will explore the mathematical concepts needed to understand the Parameter and State Estimation of random state variables of vibrating structures. We'll start with Deterministic and Stochastic Dynamics in Euclidean Space before delving into Basic Manifold theory. Then, we'll learn how to formulate dynamics in manifolds and then study applying the filters for estimation on Manifolds.

2.1 Lagrangian and Hamiltonian Dynamics on $\mathbb{R}^n$

The Lagrangian function, denoted as $L : TR^n \to \mathbb{R}^1$, is a function that takes the configurations (generalized coordinates) and their time derivatives as inputs and returns a scalar value. It is commonly used in the framework of Lagrangian mechanics to describe the dynamics of a physical system.

Mathematically, the Lagrangian function can be expressed as the difference between the system's kinetic energy and potential energy, both of which are functions of the configurations. The kinetic energy is expressed in terms of the configurations and their time derivatives, while the potential energy is expressed solely in terms of the configurations.

In equation form, the Lagrangian function can be written as:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q) \tag{2.1}$$

where:

1. L represents the Lagrangian function.
2. q denotes the configurations (generalized coordinates) of the system.
3. $\dot{q}$ represents the time derivatives of the configurations.
4. $T(q, \dot{q})$ is the kinetic energy of the system, which is a function of the configurations and their time derivatives.
5. $V(q)$ is the potential energy of the system, which is a function of the configurations.

The Lagrangian function plays a crucial role in Lagrangian mechanics, as it provides a concise and elegant formulation for deriving the equations of motion of a physical system. By applying the principle of least action, known as Hamilton's principle, the equations of motion can be derived by minimizing the action functional associated with the Lagrangian function.

From this, using the Hamilton's Variational Principle, the Euler-Lagrange Equation can be derived. The Euler-Lagrange Equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q, \dot{q})}{\partial q} = 0. \tag{2.2}$$

This gives us the Equation of Motion of the system under consideration.

We also look at the Hamiltonian Formulation as it helps us in proving the symplectic nature of our numerical scheme[6].

In systems where there is no randomness or uncertainty involved, the dynamics can be described using the Hamiltonian formalism. In this framework, the Hamiltonian and the canonical equations play a key role in determining the system's behavior.

The Hamiltonian, denoted as $\widetilde{H}(q,p)$, is a function defined on the phase space $T * \mathbb{R}^n$ which consists of the generalized coordinates q and their corresponding momenta p.

$$\widetilde{H}(q,p) = p \cdot \dot{q} - L(q,\dot{q})\Big|_{p=\frac{\partial L}{\partial \dot{q}}} \tag{2.3}$$

$$\frac{dq}{dt} = -\frac{\partial \widetilde{H}}{\partial p}, \quad \frac{dp}{dt} = \frac{\partial \widetilde{H}}{\partial q} \tag{2.4}$$

Together, the Hamiltonian and the canonical equations provide a complete and deterministic description of the dynamics in systems without any randomness or uncertainty. By solving these equations, one can determine the evolution of the system's coordinates and momenta over time.

2.2 Manifolds

2.2.1 Definition

An abstract mathematical space known as a manifold has a structure that may be more complex globally but is similar to the Euclidean geometry-described spaces locally. For instance, the surface of the Earth is varied; although locally it appears flat when seen globally from space, it is actually rounded. It is possible to 'glue' various Euclidean spaces together to create a manifold [7]. A circle S^1 is an example of a manifold. Although a small portion of a circle resembles a slightly bent portion of a straight-line segment, the circle, and the segment are actually two different 1D manifolds. A segment of a straight line can be bent, and the ends can be joined with glue to create a circle. Examples of 2D manifolds include the surfaces of a sphere and a torus. Manifolds are crucial components of mathematics, physics, and control theory because they enable the expression and comprehension of more complex structures in terms of the well-known characteristics of simpler Euclidean spaces.

Take into consideration a set M that is a potential manifold. Any point x on M has

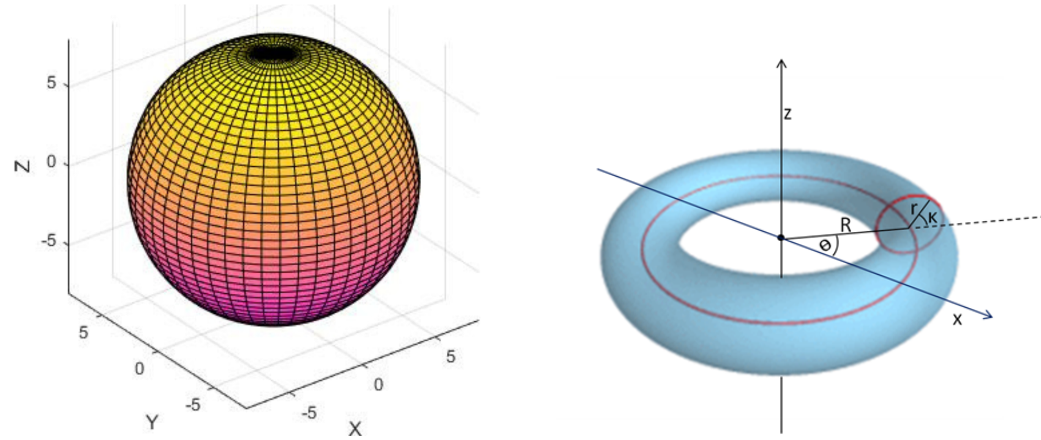

Fig. 2.1 Manifolds (a) Hypersphere (b) Torus

an associated Euclidean chart, which is given by a one-on-one mapping and plotted onto the map $\theta_i : M \to \mathbb{R}^n$, with an associated Euclidean image $V_i = \theta_i(U_i)$. Where U_i belongs to M and $V_i \in \mathbb{R}^n$.

2.2.2 Notations

There are numerous mathematical operations that arise in the analysis and operations of manifolds. So, it is a good practice to fix all the notations which will be then used to describe the mathematics in the further sections. M stands for a Riemannian manifold. The tangent space to the manifold M at a point $x \in M$ is designated as T_xM. The tangent bundle denoted by the symbol TM is defined as $TM := (x, v), x \in M, v \in T_xM$. A manifold is endowed with a local metric, this local metric defines the local norm $\|V_x\| = \sqrt{(V \cdot V^T)}$ for $V \in T_xM$. The Riemannian gradient of a function $\psi : \mathbb{R} \to M$ evaluated at any point $x \in M$ is denoted as $\text{grad}_x\psi$. The covariant derivative of a vector field $w \in TM$ in the direction of $v \in T_xM$ is denoted by $\nabla_v w$. We assume M to be endowed with a metric connection. The parallel transport operator $P(x \to y)$ transports a tangent vector from $T_xM \to T_yM$. A manifold exponential map $\exp : TM \to M$ is applied as $\exp_x(v)$. Its inverse 'log' is defined locally and is denoted by $\log_{y,x}$ where y is the reference point $y \in M$. Given two points in the manifold M, the distance between them is denoted by $d(x, y)$, which is termed as the Riemannian distance between x and y. Now, having prepared a notation for all the objects, now we define each of the terms individually.

Tensors

A tensor is an object that is invariant under a change of coordinates and has components that change in a special predictable way under the change of coordinates. A tensor is a collection of vectors and co-vectors combined together using the tensor product. Tensors follow two transformation rules:

- Forward Transformation (F):
$$T'_{ij} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l}T_{kl}$$
- Backward Transformation (B):
$$T_{kl} = \frac{\partial x_i}{\partial x'_k}\frac{\partial x_j}{\partial x'_l}T'_{ij}$$

  such that $F \cdot B = I$.

Co-Vectors

A function that takes a vector and produces a scalar is called a co-vector. Spaces of co-vectors are called dual spaces. Co-vectors are invariant but co-vector components are not invariant.

Metric Tensor

A metric tensor is an additional structure on a manifold M in the field of differential geometry that allows for the definition of distances and angles, much like the inner product on an Euclidean space does. A metric tensor on M is made up of metric tensors at each point $p \in M$ that vary smoothly with p, and a metric tensor at a point $p \in M$ is a bilinear form defined on the tangent space at p. Metric Tensor is one of the most important objects in Manifolds as it is the metric tensor that allows us to do mathematical operations, such as finding the area, shortest distance, etc in manifolds. Consider the example of the Pythagoras theorem, which finds us the hypotenuse distance in a triangle, but is only valid in the orthonormal basis, so to do this in a manifold we require a metric tensor.

There are two methods available to define the metric tensor. These are:

1. Intrinsic Approach: The manifold is not defined by the intrinsic view as being enmeshed in another space. Instead, it defines it using a metric or other relevant data that informs us of curvature. This is also referred to as the "bug-eye view".

2. Extrinsic Approach: A manifold's extrinsic view is a portion of a larger space, typically a space with more dimensions. In that case, the manifold can be described by an equation that indicates which points it takes up in the larger space. For instance, the unit sphere is seen as a subset of Euclidean 3-space when viewed extrinsically. This is also referred to as the "bird eye view".

Mathematically, a metric tensor is the inner product of the basis vectors of the manifold under consideration.

$$G = \begin{bmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{bmatrix}$$

Where, the components of the metric tensor are the inner products of the basis vectors in the intrinsic space, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. This is the intrinsic definition where the manifold under consideration is the sphere. In the case of extrinsic view, the metric tensor is defined as follows:

$$G = \begin{bmatrix} \mathbf{e}_x \cdot \mathbf{e}_x & \mathbf{e}_x \cdot \mathbf{e}_y & \mathbf{e}_x \cdot \mathbf{e}_z \\ \mathbf{e}_y \cdot \mathbf{e}_x & \mathbf{e}_y \cdot \mathbf{e}_y & \mathbf{e}_y \cdot \mathbf{e}_z \\ \mathbf{e}_z \cdot \mathbf{e}_x & \mathbf{e}_z \cdot \mathbf{e}_y & \mathbf{e}_z \cdot \mathbf{e}_z \end{bmatrix}$$

Where the **e** basis vectors are from the Euclidean space. Once this is established, we can perform the operations on geometrical surfaces. The Metric Tensor is also called the First Fundamental Form.

Geodesics

The straightest possible path we can draw on surfaces between two points is called a geodesic. A curve that minimises length locally is a geodesic. It is, in effect, the path that a particle that is not accelerating would take. The geodesics are lines that are straight in the plane. The geodesics on the sphere are large circles. The notions of distance and acceleration are impacted by the Riemannian metric, which also affects the geodesics in a space [8].

In addition to having many other intriguing qualities, geodesics maintain a direction on a surface. Any point on a geodesic arc has a normal vector that runs parallel to the surface there. The Equation of Geodesic is given by:

$$\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda} \cdot \frac{du^j}{d\lambda} = 0$$

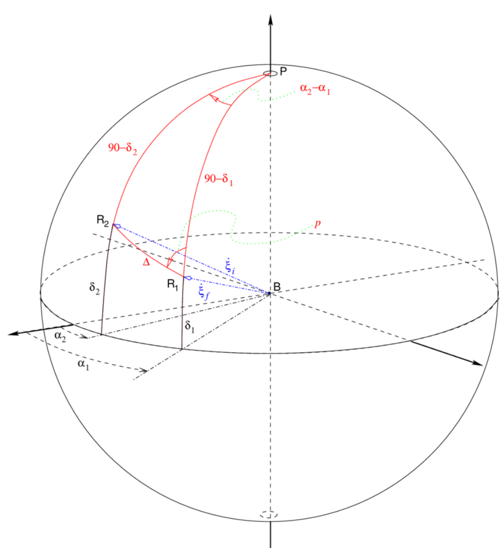

Fig. 2.2 Geodesic between two points on Sphere

Christoffel symbols

The Christoffel symbols are a set of numbers that represent the metric link between mathematics and physics. A metric can be used to measure distances on surfaces or other manifolds thanks to the metric connection, a specialization of the affine connection. The Christoffel symbols give a concrete illustration of how coordinates on the manifold relate to (pseudo-)Riemannian geometry. Then, more ideas, such as parallel transportation, geodesics, etc., can be expressed using Christoffel symbols. When there is some symmetry between the coordinate system and the metric tensor, many of the Christoffel symbols are zero. Below is a small derivation on how geodesic equation is obtained and the role of Christoffel symbols in the geodesics of a surface. Note that the Einstein's Summation notation is followed. Methods to find the geodesic equation We need to solve the equation.

$$\frac{d^2\vec{R}}{d\lambda^2} = \frac{d^2\vec{R}}{d\lambda^2}_{\text{tangential}} + \frac{d^2\vec{R}}{d\lambda^2}_{\text{normal}} \tag{2.5}$$

Now, Tangential Component = 0, on expanding the above equation we get,

$$\frac{d^2R_j}{d\lambda^2} = \left(\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\right)\frac{\partial\vec{R}}{\partial u^k} + L_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\hat{n} \tag{2.6}$$

$$\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0 \quad \text{(Geodesic Equation)} \tag{2.7}$$

$$\Gamma^k_{ij} = \frac{\partial^2\vec{R}}{\partial u^j \partial u^j} \cdot \frac{\partial\vec{R}}{\partial u^l} g^{lk} \quad \text{(Christoffel Symbol)} \tag{2.8}$$

Here u^i and u^j are the basis vectors and R is the position vector. And g^{lk} is the component of inverse metric tensor.

Covariant Derivative

The covariant derivative in mathematics is a method of expressing a derivative along tangent vectors of a manifold. In contrast to the method provided by a main connection on the frame bundle, the covariant derivative is a manner of introducing and dealing with a connection on a manifold through the use of a differential operator [17].

Derivation: Suppose we have a vector field $A(x^i)$ Writing in terms of basis vectors

$$
\begin{aligned}
dA &= d(A^i de^i) \\
&= (dA^i)e^i + A^i(de^i) \\
&= \left(\frac{\partial A^i}{\partial x^j} dx^j\right) e^i + A^i \left(\frac{\partial e^i}{\partial x^j} dx^j\right) \\
&= \left(\frac{\partial A^i}{\partial x^j} dx^j\right) e^i + A^i \Gamma^k_{ij} e^k dx^j \\
&= \left(\frac{\partial A^i}{\partial x^j} + A^i \Gamma^k_{ij}\right) e^k dx^j
\end{aligned}
$$

Parallel Transport

Parallel transport is a technique used in geometry to move geometrical information along a manifold's rounded curves. If the manifold has an affine connection, one can move the manifold's vectors along curves while maintaining their parallelism with regard to the connection. Thus, the parallel transport for a connection provides a means of connecting the geometries of close-by points, or in certain ways moving the local geometry of a manifold along a curve. There may be other parallel transport concepts, but this definition only refers to one method of joining the geometries of points on a curve. In actuality, parallel transport is the infinitesimal analog of the conventional notion of connection. The connection here is referred to as the covariant derivative.

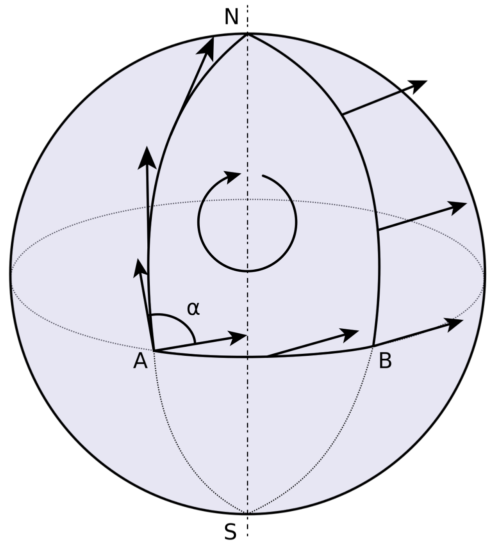

Fig. 2.3 Parallel transport of a vector in a manifold

The notion of covariant derivative is closely tied to the notion of parallel transport along a curve. The parallel transport operator $P^r_{x\to y} : T_{\gamma(x)}M \to T_{\gamma(y)}M$ associated with the curve $\gamma : I \to M$ with $\theta \in I$, $\gamma(0) = x'$, and $u, w \in T_{x'}M$ is given by

$$G_{\gamma(t)}(P^r_{0\to t}\gamma(v), P^r_{0\to t}\gamma(w)) = G_{x'}(u, w) \tag{2.9}$$

The covariant derivative of a vector field $E \in X(M)$ in the direction ω is related to the parallel transport operator by

$$\nabla_\omega E = \frac{d}{dt} P^t_{\epsilon\to 0} E(\gamma(t))\bigg|_{t=0} \tag{2.10}$$

If vector field E satisfies the condition $P^t_{\epsilon\to 0}E(\gamma(x)) = E(\gamma(y))$, the field $\vec{t}$ is said to be parallel along r. The parallelism term in terms of covariant derivative is $\nabla_{\dot\gamma}E = 0$.

A geodesic in manifold M with connection ∇ and associated parallel translation operator P_G, is a curve γ such that $\dot\gamma$ is parallel translated along r itself.

$$P^t_{Gs\to\gamma}(\dot\gamma(s)) = \dot\gamma(t) \tag{2.11}$$

Exponential Mapping

In Riemannian geometry, an exponential map refers to a mapping from a subset of the tangent space T_pM of a Riemannian manifold M to the manifold itself. This exponential map is determined by the canonical affine connection established by the Riemannian metric. The geodesic equation, given by [17]

$$\ddot{x}^k + \sum_{i,j} \Gamma^k_{ij}(\gamma)\dot{x}^i\dot{x}^j = 0 \tag{2.12}$$

describes the behavior of geodesics on the manifold. For any vector V in T_xM, there exists an interval I around the origin O and a unique geodesic $\gamma(t,x) : I \to M$ such that $\gamma(0) = x$ and $\dot{\gamma} = v$. The exponential map, denoted as $\exp : T_xM \to M$, maps each vector v in the tangent space to a point on the manifold, denoted as $\exp_x v$, given by $\gamma(1; x, v)$.

A manifold M is said to be geodesically complete if the domain of the exponential map, denoted as Exp, covers the entire tangent space T_xM for every x in M.

Lograthimc Mapping

In Riemannian geometry, it is the inverse of the exponential mapping, such that it returns a vector which belongs to the tangent space of M which is the direction vector between two point in the manifold.

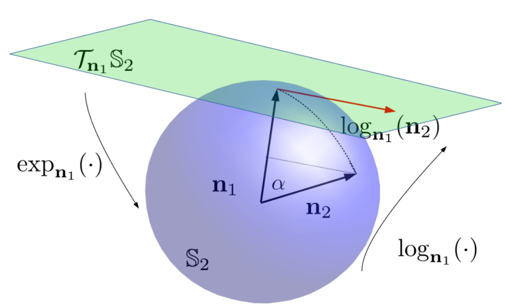

Fig. 2.4 Exponential Mapping and Logarithmic Mapping on surface of S^2 manifold

2.3 Lie Groups and Lie Algebras

A Lie group is a smooth manifold that also carries a group structure whose product and inversion operations are smooth as maps of manifolds. These structures naturally appear when describing physical symmetries.

A Lie group is a group whose elements can have a continuous real number parametrization, like the rotation group $SO(3)$, which can have the Euler angles as its parametrization. An analytic real or complex manifold that is also a group, such that the group operations multiplication and inversion are analytic maps, is referred to as a Lie group in a more formal sense.

2.3.1 Group Axioms

1. **Closure:** If $a, b \in G$, then $\phi(a, b) \in G$.
2. **Associative:** If $a, b, c \in G$, then $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$.
3. **Identity:** If $a \in G$, then there exists $e \in G$ such that $\phi(a, e) = \phi(e, a) = a$.
4. **Inverse:** If $a \in G$, then there exists a unique element $a^{-1} \in G$ such that $\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$.

2.3.2 Lie Group

A Lie group is a smooth manifold M with a group G structure that is concurrently consistent with its manifold M structure in the sense of group multiplication and group inversion. The group identity element is a point e at G. Every point of the manifold in a Lie group has the same appearance, so every tangent space at every point is the same. The group structure mandates that each element's composition stays on the manifold, and that each element also has an inverse in the manifold. Calculus on groups is possible thanks to Lie groups, which connect the local characteristics of smooth manifolds.

2.3.3 Group Actions

Lie groups possess the ability to alter the elements of other sets, leading to transformations such as rotations, translations, scaling, and combinations thereof. For a valid group action, it must satisfy the axioms of identity and compatibility.

2.3.4 Lie Algebra

If we have a point $X(t)$ that moves on a manifold M associated with a Lie group, the velocity of this point belongs to the tangent space of the manifold, denoted as T_xM. The smoothness of the manifold ensures the presence of a unique tangent space at each point. The tangent space at the identity element of the Lie group is known as the Lie algebra of that particular group. It is important to note that every Lie group is accompanied by its own corresponding Lie algebra.

2.3.5 Exponential Map and Logarithmic Map

The exponential map is a mapping that establishes a diffeomorphism between the Lie algebra and the Lie group. It allows us to convert elements from the Lie algebra to corresponding elements in the Lie group. Conversely, the inverse of the exponential map is known as the logarithmic map, which enables us to go from the Lie group back to the Lie algebra.

In our research, calculating the exponential mapping directly poses challenges as it necessitates a deep understanding of advanced concepts in Differential Geometry. As a result, we opt for a shortcut by leveraging the action exerted by Lie groups on the configuration manifold considered in our study. This approach provides us with a more accessible way to incorporate the effects of Lie groups in our analysis.

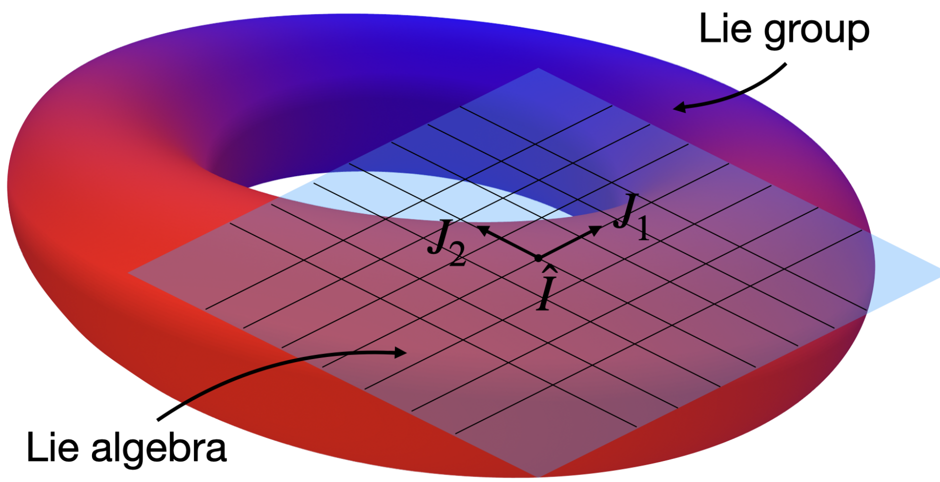

Fig. 2.5 Working of Lie Groups and Lie Algebra

Rotational Lie Groups

This group is the subset of the General Linear Group whose Group Action represents the rotations induced in any physical system. The Rotational Groups are $n \times n$ matrices where n is the number of rotation axes. The Rotational group may be of n dimensions, but the rotations in 2 and 3 dimensions are important.

Uni-axial Rotation Groups or $SO(2)$

The following transformation of the joint coordinates results from the uniaxial joint rotation in a single Cartesian plane around a perpendicular axis, for example, the $x - y$ plane about the z axis with rotation angle θ:

$$SO(2) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Lie Algebra:

$$\mathfrak{so}(2) = \left\{ \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} , \quad t \in \mathbb{R} \right.$$

Exponential Map:

$$\exp\left(\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}\right) = \gamma_\theta(1) = \begin{bmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{bmatrix}$$

Three-axial Rotation Groups or $SO(3)$

The group $SO(3)$ is made up of rotation matrices or special orthogonal matrices in 3D space that are subject to matrix multiplication. In all groups SO, inversion and composition are accomplished through transposition and product (n). The lie algebra of the group is defined by angular velocities ω_x, ω_y, ω_z.

$$[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Exponential Map:

$$R = \exp([\omega]\theta) \in SO(3), \text{ where } R \text{ is the rotation matrix.}$$

$$R = I + [\omega]\sin(\theta) + [\omega]^2(1 - \cos(\theta))$$

Logarithm:

$$\theta[\omega] = \log(R)\theta(R - R^\top)\frac{1}{2\sin(\theta)}$$

Where,

$$\theta = \cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right)$$

2.4 Kalman Filter for Estimation on Manifolds

The Kalman filter is a recursive algorithm that estimates the state of a dynamic system from a series of noisy measurements. It is widely used in various fields, including control systems and robotics, for state estimation. When dealing with systems evolving on manifolds, such as rotations or orientations, a modified version of the Kalman filter, known as the Extended Kalman Filter (EKF) or the Unscented Kalman Filter (UKF), is often employed.

2.4.1 Mathematical Background

Consider a dynamic system evolving on a manifold M with state $x \in M$. The state evolves according to a dynamic model, and measurements z related to the state are obtained with noise. The Kalman filter formulates the state estimation problem using the following equations:

State Prediction

The prediction of the state is given by the system dynamics:

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_{k-1})$$

where f is the state transition function, $\hat{x}_{k|k-1}$ is the predicted state, $\hat{x}_{k-1|k-1}$ is the previous state estimate, and u_{k-1} is the control input.

Error Covariance Prediction

The error covariance matrix is predicted using the Jacobian of the state transition function:

$$P_{k|k-1} = A_{k-1} P_{k-1|k-1} A_{k-1}^{\top} + Q_{k-1}$$

where A_{k-1} is the Jacobian of f with respect to the state, $P_{k-1|k-1}$ is the error covariance matrix of the previous estimate, and Q_{k-1} is the process noise covariance.

Measurement Prediction

The predicted measurement is obtained using the measurement model:

$$\hat{z}_{k|k-1} = h(\hat{x}_{k|k-1})$$

where h is the measurement function.

Kalman Gain Calculation

The Kalman gain is computed to determine the weight of the measurement in the state correction:

$$K_k = P_{k|k-1} H_k^\top (H_k P_{k|k-1} H_k^\top + R_k)^{-1}$$

where H_k is the Jacobian of the measurement function, and R_k is the measurement noise covariance.

State Update

The state is updated based on the measurement and the Kalman gain:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(z_k - \hat{z}_{k|k-1})$$

Error Covariance Update

The error covariance matrix is updated using the Kalman gain:

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

where I is the identity matrix.

2.4.2 Application to Manifolds

When dealing with state spaces that are manifolds, the standard Kalman filter formulation may not be directly applicable due to the non-Euclidean nature of the manifold. In such cases, the Extended Kalman Filter (EKF) or the Unscented Kalman Filter (UKF) is often used to linearize or approximate the manifold.

The EKF linearizes the state transition and measurement functions, allowing the use of the standard Kalman filter equations. The UKF, on the other hand, avoids linearization by approximating the probability distribution of the state using a set of carefully chosen sample points, known as sigma points.

For example, when dealing with rotational manifolds, such as $SO(3)$, the state space can be parametrized using unit quaternions, and the EKF or UKF can be applied to estimate the quaternion states.

The choice between EKF and UKF depends on the specific characteristics of the manifold and the desired trade-off between accuracy and computational complexity.

In summary, the Kalman filter and its variants provide powerful tools for state estimation on manifolds, with the choice of the specific variant depending on the nature of the manifold and the application requirements.

2.5 Summary

In this chapter, we covered the fundamental concepts necessary for grasping the essentials of Manifolds and Kalman Filter. Additionally, we explored the workings of Lie Groups, particularly focusing on Rotational Lie Groups, which will play a crucial role in our formulation. Prior to delving into the formulation itself, we will establish the methodology for expressing dynamics on manifolds. The subsequent chapter will introduce the formulation of filter, with a specific emphasis on the Spherical Manifold.

Chapter 3

Estimation of Spherical Pendulum Dynamics using Unscented Kalman Filter on Manifolds

3.1 Introduction

In this chapter, we explore the application of the Unscented Kalman Filter (UKF) on parallelizable manifolds for the estimation of the dynamics of a spherical pendulum[9]. The spherical pendulum serves as a representative example of systems evolving on manifolds, particularly the two-sphere manifold.

3.2 Problem Formulation

The set of all points in the Euclidean space $\mathbb{R}^3$, lying on the surface of the unit ball about the origin, belongs to the two-sphere manifold, denoted as $S^2 = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$. Systems such as a spherical pendulum evolve on the two-sphere manifold.

3.3 Simulation of Spherical Pendulum Dynamics

We initiated our simulation by defining the parameters of the system and setting up the dynamic equation for the spherical pendulum on the special orthogonal group $SO(3)$ manifold. The model parameters included the total simulation time T, model frequency f, and model noise standard deviation $model\_noise\_std$. We considered a spherical pendulum with the following characteristics:

- Length of the wire (L): 1.3 meters
- Gravity constant (g): $9.81\,\mathrm{m/s^2}$

The integration step dt was determined based on the chosen frequency to ensure accurate numerical simulation.

3.3.1 Data Generation and Trajectory Simulation

We generated simulated data representing the behavior of the spherical pendulum system over time. The simulation involved the following key steps:

1. **Initialization:** We initialized the simulation environment and defined the model parameters.

2. **Model and Simulation:** We employed the Geometric Ito-Taylor 1.5 method to numerically simulate the true states and noisy inputs of the spherical pendulum system. The simulation incorporated model noise to account for uncertainties and external disturbances.

3. **Plotting:** We visualized the trajectory of the spherical pendulum on the two-sphere manifold S^2 and overlaid it on a spherical surface. The simulated trajectory represented the motion of the pendulum over the specified simulation time.

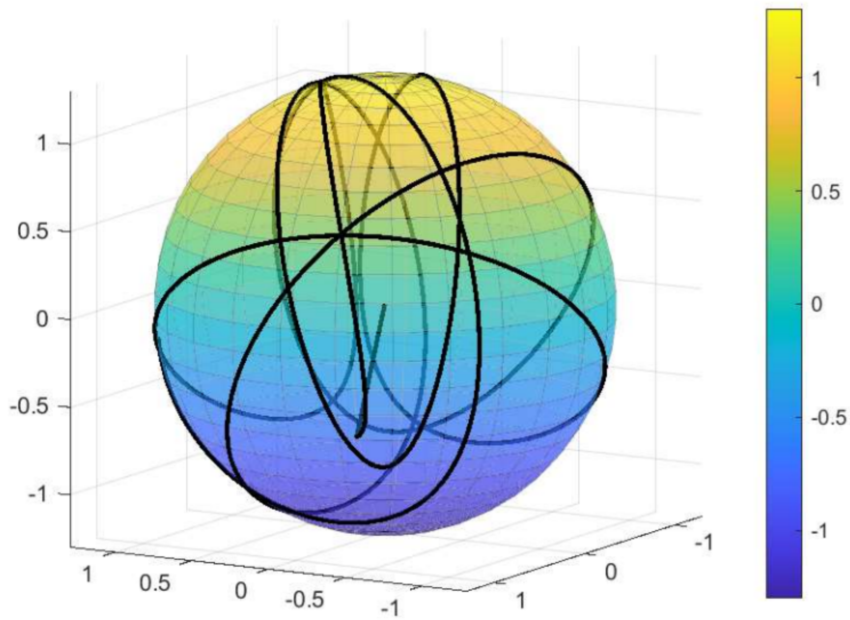

Fig. 3.1 Trajectory on S03 Manifold

3.4 Simulation Results

In this section, we present the results of our MATLAB simulation of a spherical pendulum equipped with a geometrically consistent tuned mass damper (TMD) using the Geometric Ito-Taylor 1.5 method. We explore the behavior of the system on the two-sphere manifold S^2 and analyze its response in terms of orientation and position over time. The simulation results demonstrate the dynamic behavior of a spherical pendulum equipped with a geometrically consistent tuned mass damper. Several key observations can be made from the generated data:

1. **Spherical Pendulum Motion:** The trajectory of the spherical pendulum on S^2 illustrates the complex and chaotic motion of the system. The motion includes rotations and oscillations, showcasing the nonlinear nature of the dynamics.

2. **Effect of Model Noise:** The inclusion of model noise in the simulation accounts

for real-world uncertainties and disturbances. This noise influences the pendulum's motion, introducing variations in orientation and position.

3. **Tuned Mass Damper:** The simulation can incorporate the behavior of a geometrically consistent tuned mass damper (TMD). The TMD's role is to mitigate excessive structural motions, and its effectiveness can be analyzed by comparing the system's response with and without the TMD.

Overall, this simulation serves as a valuable tool for understanding the behavior of complex mechanical systems like spherical pendulums equipped with geometrically consistent TMDs. It provides insights into the system's response to external forces and disturbances and can aid in the development and optimization of TMD strategies for structural control.

3.5 Model and Simulation

We utilize the UKF-M methodology, a novel approach for implementing the Unscented Kalman Filter on parallelizable manifolds. The model is based on the Euler equations of pendulum motion. The simulation involves generating true states and noisy inputs, as well as simulating landmark measurements based on the true states.

3.6 Filter Design and Initialization

The state of the system is embedded in $SO(3) \times \mathbb{R}^3$ with left multiplication. The propagation noise covariance matrix, measurement noise covariance matrix, and initial uncertainty matrix are appropriately defined. The UKF is initialized with the chosen parameters and state.

3.7 Filtering

The UKF proceeds with a standard Kalman filter loop, involving state propagation and update steps based on received measurements. The estimates of the state and covariance are recorded along the trajectory.

3.8 Results

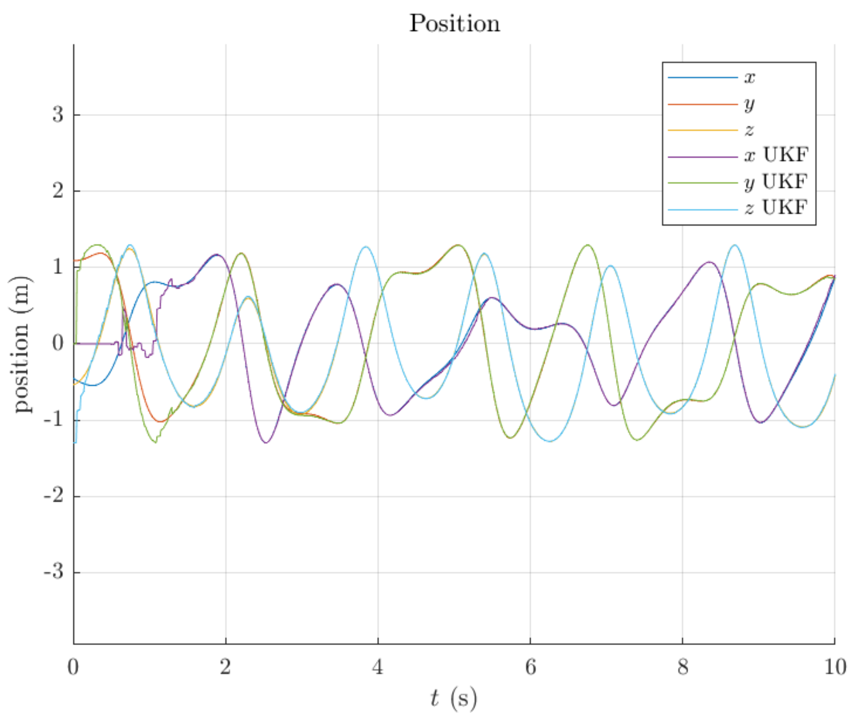

Fig. 3.2 Position(m)

The results showcase the accuracy, robustness, and consistency of the UKF in estimating the position of the spherical pendulum, even in the presence of strong initial errors. Plots depict the position of the pendulum as a function of time, along with 3σ interval confidence, demonstrating the convergence to the true state and the consistency of the filter.

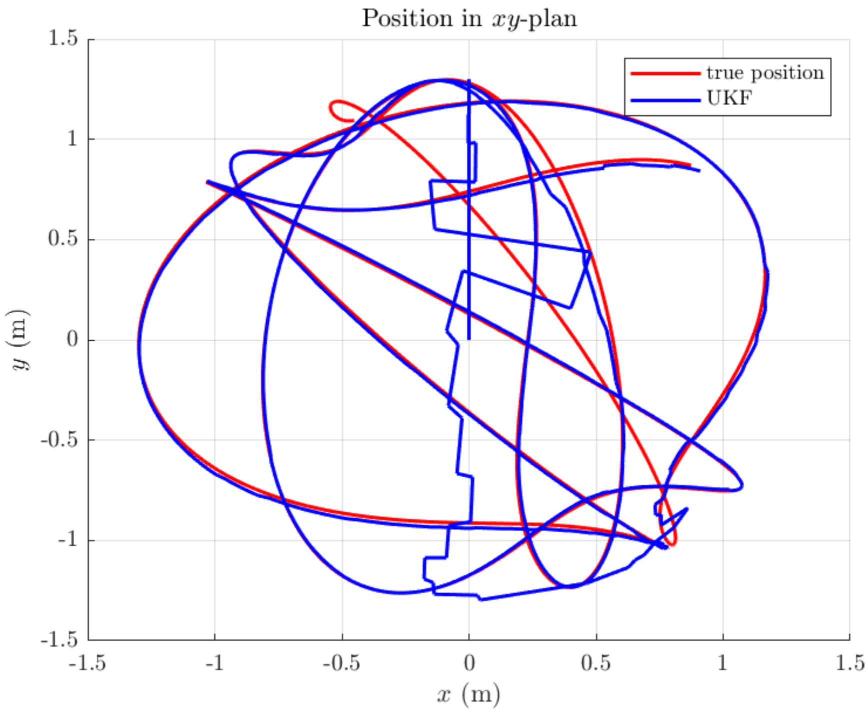

Fig. 3.3 Position of Pendulum in XZ Plane

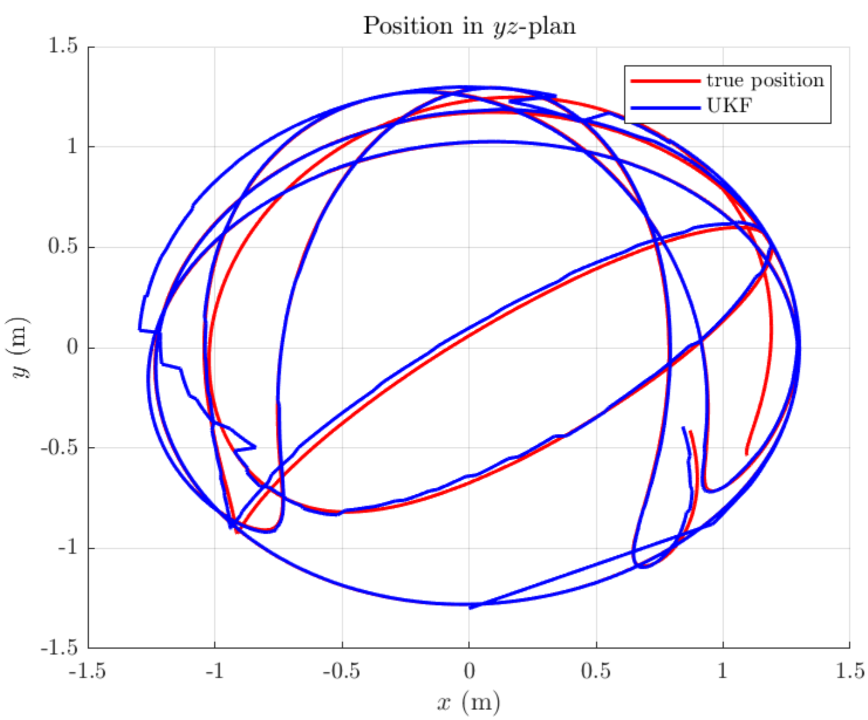

Fig. 3.4 Position of Pendulum in XZ Plan

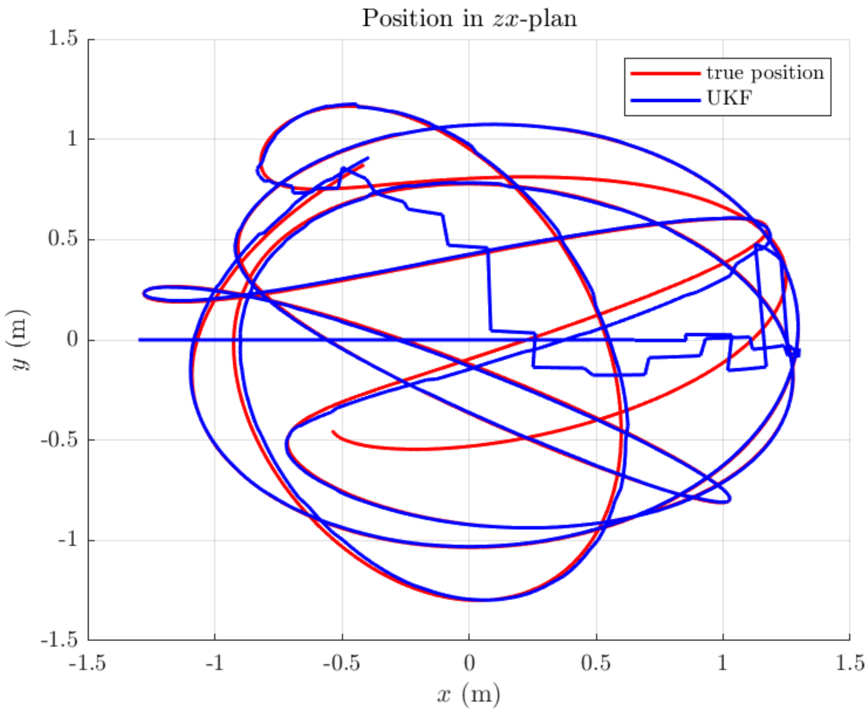

Fig. 3.5 Position Error

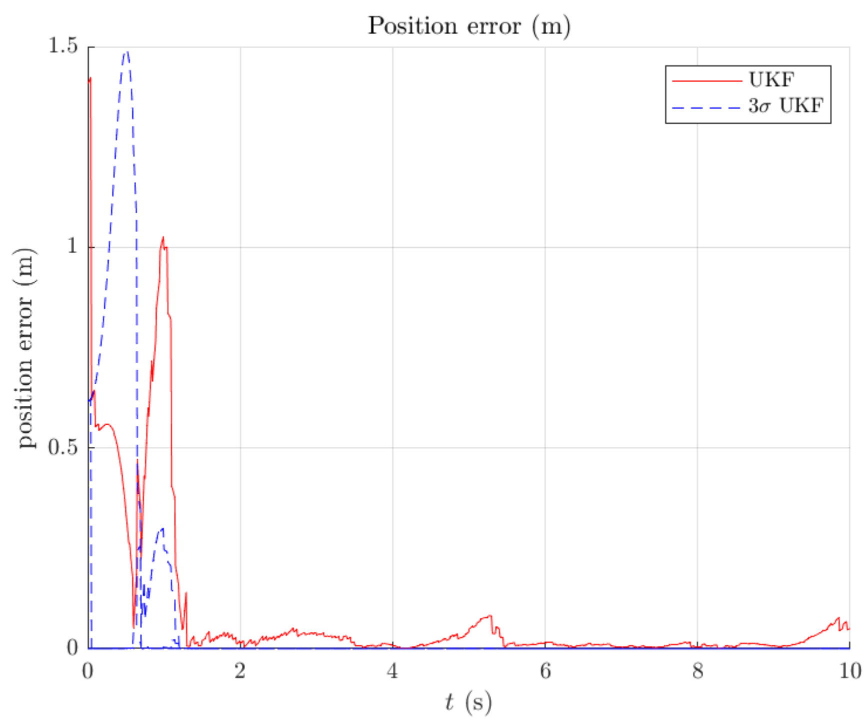

Fig. 3.6 Position Error

3.9 Conclusion

This chapter demonstrates the successful application of the UKF on parallelizable manifolds for estimating the position of a spherical pendulum. The filter exhibits accuracy, robustness, and consistency, laying the groundwork for further exploration and application in diverse scenarios.

Chapter 4

Conclusion and Future Work

The exploration of state and parameter estimation on manifolds using Lie groups has opened up exciting avenues for future research and practical applications, especially in the realms of structural engineering and control. This chapter outlines potential future directions and use cases, pointing towards the continued evolution and application of this innovative approach.

4.1 Advanced State Estimation Techniques

Future research endeavors can delve into the development of advanced state estimation techniques, further harnessing the capabilities of the Lie group framework. This entails refining algorithms and methodologies for precise estimation of dynamic system states residing on manifolds such as the Special Orthogonal Group $SO(3)$ or other Lie groups. Improved accuracy and robustness in state estimation can empower more effective control strategies for intricate systems.

4.2 Parameter Estimation on Manifolds

Extending state estimation to parameter estimation stands out as a promising avenue for future exploration. Structural systems often feature parameters with variability or uncertainty, such as mass distributions, material properties, and damping coefficients. Future research can focus on developing methods that concurrently estimate both the system state and its parameters, taking into account the inherent manifold structure. This holistic approach enhances our ability to model and control complex systems effectively.

4.3 Retuning Strategies

A practical application of state and parameter estimation on manifolds lies in the field of detuning – the adjustment of system parameters, such as damping properties of tuned mass dampers, to optimize structural performance. Subsequent research can investigate how Lie group-based estimation techniques contribute to adaptively retuning systems, enabling them to counteract changing environmental conditions and ensuring structural stability and safety.

4.4 Adaptive Tuning of TMDs

The concept of adaptive tuning involves dynamically adjusting the parameters of a structure or its control systems in response to varying loads or external factors. Lie group-based estimation is poised to play a pivotal role in developing adaptive tuning strategies for structures equipped with tuned mass dampers. These strategies can optimize damper performance in real-time, reducing structural vibrations and enhancing overall system efficiency.

4.5 Health Monitoring and Maintenance

State and parameter estimation techniques on manifolds find applications in health monitoring and maintenance of structures. Continuous monitoring of a structure's state and parameters, including those of tuned mass dampers, facilitates the detection of anomalies and degradation. This proactive approach to maintenance can significantly extend the lifespan of structures, bolstering safety and structural integrity.

4.6 Conclusion

As we embark on these future research directions and applications, the integration of Lie groups into state and parameter estimation methodologies promises to revolutionize our approach to structural engineering and control. These endeavors hold the potential to create more resilient, adaptive, and efficient systems across various domains.

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